The Frobenius Determinant

and a Foray into Linear Representations

Dylan Wallace, adv. Kevin Dao

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WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

The Frobenius Determinant Problem

Let G be a finite group with $|G| = n$, $g_1, \ldots, g_n \in G$.

Let $x_{g_1}, \ldots, x_{g_n}$ be distinct variables indexed by elements in G, and consider the matrix X_G with entries $(X_G)_{ij} = x_{g_i g_j}$. In this case, $x_{g_i g_j} = x_{g_k}$ if $g_i g_j = g_k$.

Frobenius' Determinant Theorem

$\det X_G = \prod_{j=1}^{r} P_j(\mathbf{x})^{\deg P_j}$, where $P_j(\mathbf{x})$ are pairwise nonproportional irreducible polynomials and r is the number of conjugacy classes of G

- Pairwise-nonproportional: $P_i(\mathbf{x})$ and $P_j(\mathbf{x})$ are not scalar multiples of each other
- Irreducible: $P_j(\mathbf{x})$ can't be written as a product of two polynomials
- Conjugacy class: Disjoint sets $C \subset G$ s.t. $hgh^{-1} \in C$ for $g \in C$, $h \in G$

Example

Let $G = \mathbb{Z}/2\mathbb{Z}$, with $g_1 = \overline{0}$ and $g_2 = \overline{1}$. Then $g_1^2 = g_1$, $g_2^2 = g_1$, and $g_1 g_2 = g_2 g_1 = g_2$; the conjugacy classes of G are $\{g_1\}$ and $\{g_2\}$, so $r = 2$.

Then $X_G = \begin{bmatrix} x_{g_1} & x_{g_2} \\ x_{g_2} & x_{g_1} \end{bmatrix}$, so $\det X_G = x_{g_1}^2 - x_{g_2}^2 = (x_{g_1} + x_{g_2})(x_{g_1} - x_{g_2})$

So we have a product of

- 2 polynomials
- that are not scalar multiples of each other
- which aren't products of other polynomials

so our theorem holds for $G = \mathbb{Z}/2\mathbb{Z}$.

It turns out we can prove this theorem in the general case with **Representation Theory**.

Basic Representation Theory

Representation Theory is the study of group actions on vector spaces.

Definition

Given a group G and vector space V, a **representation** is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. The **degree** of ρ is $\dim V$.

- We will only be looking at representations of finite groups ($|G| = n$ for the rest of this presentation)
- We will be looking at finite representations over $\mathbb{C}$. In other words, $\dim V < \infty$ always. This keeps things clean and nice

Examples of representations

- $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ with $\rho(\overline{0}) = 1$, $\rho(\overline{1}) = -1$ is a representation of degree 1
- More generally, if $\zeta^n = 1$, then $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ with $\rho(\overline{k}) = \zeta^k$ is a representation of degree 1
  - ρ essential permutes the n-th roots of unity.
- $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^2)$ by $\rho(r) = \begin{bmatrix} e^{2\pi/3} & 0 \\ 0 & e^{4\pi/3} \end{bmatrix}$ and $\rho(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a representation of degree 2

Group Algebra

The **group algebra** of G is the ring $\mathbb{C}[G] = \left\{ \sum_{i=1}^{n} a_i g_i : a_i \in \mathbb{C} \right\}$.

- We can think of $\mathbb{C}[G]$ as a vector space with $g_1, \ldots, g_n \in G$ as bases
- Then G acts on $\mathbb{C}[G]$ by translation: $h\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i h g_i$ for $h \in G$

Note that $\mathbb{C}[G]$-modules are precisely $\mathbb{C}$-representations of G: Suppose $(V, +, \cdot)$ is a $\mathbb{C}[G]$-module. Then we can construct a representation $\rho : G \to \mathrm{GL}(V)$ by $\rho(g) : \vec{v} \mapsto g \cdot \vec{v}$. In this way, we have an equivalence between $\mathbb{C}[G]$-modules and representations of G over $\mathbb{C}$.

We specifically consider $\mathbb{C}[G]$ as a module over itself, defined by left-translation. From this we can define $\rho : G \to \mathrm{GL}(\mathbb{C}[G])$ by $\rho(g)h = gh$, $\forall g, h \in G$.

This representation is called the **regular representation**.

Reducible and Irreducible Representations

Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and let $W \subseteq V$ be a subspace.

Definition

W is G-**invariant** under ρ if $\rho(g)W = W$ for all $g \in G$.

Because we are operating over $\mathbb{C}$, there exist many 1-dimensional subspaces that are invariant under $\rho(g)$. But the span of an eigenvector is G-invariant **only if it is an eigenvector of each** $\rho(g)$!

Definition

A representation $\rho : G \to \mathrm{GL}(V)$ is **reducible** if there exist G-invariant subspaces $W, W' \subset V$ s.t. $V = W \oplus W'$.
If the only subspaces that are G-invariant under ρ are 0 and V itself, then ρ is **irreducible**.

Important Theorems

Schur's Lemma

Let $\rho : G \to \mathrm{GL}(V)$, $\rho' : G \to \mathrm{GL}(W)$, and $f : V \to W$ with $f \circ \rho(g) = \rho'(g) \circ f$ for all $g \in G$. Then

- if $\rho \not\simeq \rho'$, then $f = 0$;
- if $V = W$ and $\rho = \rho'$, then f is a scalar product (a "homothety")

Note: $\rho \simeq \rho'$ if there exists some isomorphism $f : V \to W$ s.t. $f \circ \rho(g) = \rho'(g) \circ f$.

From ρ, ρ', and **any** map $h : V \to W$, we can construct a map f s.t. $f \circ \rho(g) = \rho'(g) \circ f \implies$ we can use this map to see if any two representations are isomorphic

Important Theorems contd.

Maschke's Theorem

If k is a field whose characteristic doesn't divide $|G|$, then $k[G] \simeq \bigoplus_i \operatorname{End} V_i$, where V_i are the irreducible representations of G over k.

- Characteristic of $\mathbb{C}$ is 0 $\implies$ we can use Maschke's Theorem
- $\operatorname{End} V_i \simeq \bigoplus_{\dim V_i} V_i$ and $\dim \mathbb{C}[G] = n$, so $n = \sum_i (\dim V_i)^2$
- The regular representation $G \to \mathrm{GL}(\mathbb{C}[G])$ can be decomposed into $\dim V_i$ copies of a representation $G \to \mathrm{GL}(V_i)$ for each irreducible representation V_i

thus the regular representation **contains every irreducible representation, sometimes multiple times**

Basic Idea of Frobenius' Determinant Theorem

Proof Sketch:

- Construct a polynomial $L(\mathbf{x}) = \sum_{i=1}^{n} x_{g_i} \rho(g)$. This is a matrix with the variables x_{g_i} as coefficients, and $\det L(\mathbf{x}) = \pm \det X_G$
  - If our condition holds for $L(\mathbf{x})$, it holds for X_G
- Using Maschke's Theorem, we can show that $\det L(\mathbf{x}) = \prod_{i=1}^{r} (\det L(\mathbf{x})|_{V_i})^{\dim V_i}$
- Show that each $\det L(\mathbf{x})|_{V_i}$ is an irreducible polynomial, and that the matrix coefficients of $L(\mathbf{x})|_{V_i}$ and $L(\mathbf{x})|_{V_j}$ are different ☺

Some Other Big Ideas

A lot of the plumbing between Schur's Lemma and Maschke's Theorem is done by **characters**.

Definition

Given a representation $\rho : G \to \mathbf{GL}(V)$, the **character** $\chi(g)$ of $g \in G$ is defined as $\chi(g) = \mathbf{Tr}(\rho(g))$.

- $\mathrm{Tr}(A) = \mathrm{Tr}(BAB^{-1})$ for any matrices A, B, so $\chi(g) = \chi(hgh^{-1})$ for any $g, h \in G \implies \chi$ is defined on the **conjugacy classes** of G
- $\chi(g) = \mathrm{Tr}(\rho(g)) = \lambda_1 + \cdots + \lambda_k$, where λ_i are eigenvalues of $\rho(g) \in \mathrm{GL}(V)$. Then $\chi(g^2) = \lambda_1^2 + \cdots + \lambda_k^2$, etc.
  - Thus $\chi(g^m)$, $m < |g|$ contains all the information of the eigenvalues of $\rho(g)$!